Correct answers are ticked in green.

CSS

GS-2014

(Computer & Systems Sciences)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **COMPUTER & SYSTEMS SCIENCES - December 8, 2013**
Duration : Three hours (3 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.

3. This question paper consists of three (3) parts. **Part-A** contains twenty (20) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain twenty (20) questions each, directed towards candidates for (B) Computer Science and (C) Systems Science (including Communications & Math Finance), respectively. **STUDENTS MAY ATTEMPT EITHER PART-B OR PART-C. In case, a student attempts both Parts B & C (no extra time will be given) and qualifies for interview in both B & C, he/she will have opportunity to be interviewed in both areas.** All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks.

4. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

Part A
Common Questions

1. Consider the reactions

$$X + 2Y \to 3Z$$
$$2X + Z \to Y.$$

Let n_X, n_Y, n_Z denote the numbers of molecules of chemicals X, Y, Z in the reaction chamber. Then which of the following is conserved by both reactions?

(a) $n_X + n_Y + n_Z$.

(b) $n_X + 7n_Y + 5n_Z$. ✓

(c) $2n_X + 9n_Y - 3n_Z$.

(d) $3n_X - 3n_Y + 13n_Z$.

(e) None of the above.

2. A body at a temperature of 30 Celsius is immersed into a heat bath at 0 Celsius at time $t = 0$. The body starts cooling at a rate proportional to the temperature difference. Assuming that the heat bath does not change in temperature throughout the process, calculate the ratio of the time taken for the body to reach 1 Celsius divided by the time taken for the body to reach 5 Celsius.

(a) $\log 5$.

(b) $\frac{\log 29}{\log 25}$.

(c) e^5.

(d) $1 + \log_6 5$. ✓

(e) None of the above.

3. The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and for all integers $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Then which of the following statements is FALSE?

(a) $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$ for any integer $n \geq 0$.

(b) $F_{n+2} \geq \phi^n$ for any integer $n \geq 0$, where $\phi = (\sqrt{5} + 1)/2$ is the positive root of $x^2 - x - 1 = 0$.

(c) F_{3n} is even, for every integer $n \geq 0$.

(d) F_{4n} is a multiple of 3, for every integer $n \geq 0$.

(e) F_{5n} is a multiple of 4, for every integer $n \geq 0$. ✓

4. Consider numbers greater than one that satisfy the following properties:

(a) they have no repeated prime factors;

(b) for all primes $p \geq 2$, p divides the number if and only if $p - 1$ divides the number.

The number of such numbers is

(a) 0.

(b) 5.

(c) 100.

(d) Infinite.

(e) None of the above. ✓

5. The rules for the University of Bombay five-a-side cricket competition specify that the members of each team must have birthdays in the same month. What is the minimum number of mathematics students needed to be enrolled in the department to guarantee that they can raise a team of students?

(a) 23

(b) 91

(c) 60

(d) 49 ✓

(e) None of the above.

6. Karan tells truth with probability 1/3 and lies with probability 2/3. Independently, Arjun tells truth with probability 3/4 and lies with probability 1/4. Both watch a cricket match. Arjun tells you that India won, Karan tells you that India lost. What probability will you assign to India's win?

(a) 1/2

(b) 2/3

(c) 3/4

(d) 5/6

(e) 6/7 ✓

7. Consider a sequence of non-negative numbers $\{x_n : n = 1, 2, \ldots\}$. Which of the following statements cannot be true?

(a) $\sum_{n=1}^{\infty} x_n = \infty$ and $\sum_{n=1}^{\infty} x_n^2 = \infty$.

(b) $\sum_{n=1}^{\infty} x_n = \infty$ and $\sum_{n=1}^{\infty} x_n^2 < \infty$.

(c) $\sum_{n=1}^{\infty} x_n < \infty$ and $\sum_{n=1}^{\infty} x_n^2 < \infty$.

(d) $\sum_{n=1}^{\infty} x_n \leq 5$ and $\sum_{n=1}^{\infty} x_n^2 \geq 25$.

(e) $\sum_{n=1}^{\infty} x_n < \infty$ and $\sum_{n=1}^{\infty} x_n^2 = \infty$. ✓

8. All that glitters is gold. No gold is silver.

Claims:
1. No silver glitters.
2. Some gold glitters.

Then, which of the following is TRUE?

(a) Only claim 1 follows.

(b) Only claim 2 follows.

(c) Either claim 1 or claim 2 follows but not both.

(d) Neither claim 1 nor claim 2 follows.

(e) Both claim 1 and claim 2 follow.

9. Solve $\min\ x^2 + y^2$
subject to

$$\begin{aligned} x + y &\geq 10, \\ 2x + 3y &\geq 20, \\ x &\geq 4, \\ y &\geq 4. \end{aligned}$$

(a) 32

(b) 50

(c) 52

(d) 100

(e) None of the above

10. A person went out between 4pm and 5pm to chat with her friend and returned between 5pm and 6pm. On her return, she found that the hour-hand and the minute-hand of her (well-functioning) clock had just exchanged their positions with respect to their earlier positions at the time of her leaving. The person must have gone out to chat at

(a) Twenty five minutes past 4pm.

(b) Twenty six and 122/143 minutes past 4pm.

(c) Twenty seven and 1/3 minutes past 4pm.

(d) Twenty eight minutes past 4pm.

(e) None of the above.

11. A large community practices birth control in the following peculiar fashion. Each set of parents continues having children until a son is born; then they stop. What is the ratio of boys to girls in the community if, in the absence of birth control, 51% of the babies are born male?

(a) 51:49

(b) 1:1

(c) 49:51

(d) 51:98

(e) 98:51

12. Let $f(x) = 2^x$. Consider the following inequality for real numbers a, b and $0 < \lambda < 1$:

$$f(\lambda a + b) \leq \lambda f(a) + (1-\lambda) f\left(\frac{b}{1-\lambda}\right).$$

Consider the following 3 conditions:
(1) $\lambda = 0.5$
(2) $0 < a \leq 2,\ b > 0$
(3) $a/\lambda > 2,\ 0 < b \leq 1 - \lambda$

Which of the following statements is TRUE?

(a) The above inequality holds under conditions (1) and (2) but not under condition (3).

(b) The above inequality holds under conditions (2) and (3) but not under condition (1).

(c) The above inequality holds under conditions (1) and (3) but not under condition (2).

(d) The above inequality holds under all the three conditions.

(e) The above inequality holds under none of the three conditions.

13. Let L be a line on the two dimensional plane. L's intercepts with the X and Y axes are respectively a and b. After rotating the co-ordinate system (and leaving L untouched), the new intercepts are a' and b' respectively. Which of the following is TRUE?

(a) $\frac{1}{a} + \frac{1}{b} = \frac{1}{a} + \frac{1}{b}$.

(b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{a'^2} + \frac{1}{b'^2}$.

(c) $\frac{b}{a^2} + \frac{a}{b^2} = \frac{b'}{a'^2} + \frac{a}{b'^2}$.

(d) $\frac{b}{a} + \frac{a}{b} = \frac{b'}{a'} + \frac{a'}{b'}$.

(e) None of the above.

14. Let m and n be any two positive integers. Then, which of the following is FALSE?

(a) $m+1$ divides $m^{2n}-1$.

(b) For any prime p, $m^p \equiv m\ (\text{mod} p)$.

(c) If one of m, n is prime, then there are integers x, y such that $mx + ny = 1$.

(d) If $m < n$, then $m!$ divides $n(n-1)(n-2)\cdots(n-m+1)$.

(e) If $2^n - 1$ is prime, then n is prime.

15. Consider the following statements:

(1) $b_1 = \sqrt{2}$, series with each $b_i = \sqrt{b_{i-1} + \sqrt{2}}$, $i \geq 2$, converges.

(2) $\sum_{i=1}^{\infty} \frac{\cos(i)}{i^2}$ converges.

(3) $\sum_{i=0}^{\infty} b_i$ converges if $\lim_{i\to\infty} \frac{|b_{i+1}|}{|b_i|} < 1$.

Which of the following is TRUE?

(a) Statements (1) and (2) but not (3).

(b) Statements (2) and (3) but not (1).

(c) Statements (1) and (3) but not (2).

(d) All the three statements.

(e) None of the three statements.

16. Let $x_0 = 1$ and

$$x_{n+1} = \frac{3 + 2x_n}{3 + x_n}, \quad n \geq 0.$$

$x_\infty = \lim_{n\to\infty} x_n$ is

(a) $(\sqrt{5} - 1)/2$

(b) $(\sqrt{5} + 1)/2$

(c) $(\sqrt{13} - 1)/2$

(d) $(-\sqrt{13} - 1)/2$

(e) None of the above

17. A fair dice (with faces numbered $1, \ldots, 6$) is independently rolled repeatedly. Let X denote the number of rolls till an even number is seen and let Y denote the number of rolls till 3 is seen. Evaluate $E(Y|X = 2)$.

(a) $6\frac{5}{6}$

(b) 6

(c) $5\frac{1}{2}$

(d) $6\frac{1}{3}$

(e) $5\frac{2}{3}$

18. We are given a collection of real numbers where a real number $a_i \neq 0$ occurs n_i times. Let the collection be enumerated as $\{x_1, x_2, ..., x_n\}$ so that $x_1 = x_2 = \cdots = x_{n_1} = a_1$ and so on, and $n = \sum_i n_i$ is finite. What is

$$\lim_{k \to \infty} \left(\sum_{i=1}^{n} \frac{1}{|x_i|^k} \right)^{-1/k} ?$$

(a) $\max_i(n_i|a_i|)$

(b) $\min_i |a_i|$

(c) $\min_i(n_i|a_i|)$

(d) $\max_i |a_i|$

(e) None of the above

19. Consider the following random function of x

$$F(x) = 1 + Ux + Vx^2 \mod 5,$$

where U and V are independent random variables uniformly distributed over $\{0, 1, 2, 3, 4\}$. Which of the following is FALSE?

(a) $F(1)$ is uniformly distributed over $\{0, 1, 2, 3, 4\}$.

(b) $F(1), F(2)$ are independent random variables and both are uniformly distributed over $\{0, 1, 2, 3, 4\}$.

(c) $F(1), F(2), F(3)$ are independent and identically distributed random variables.

(d) All of the above.

(e) None of the above.

20. Consider the equation $x^2 + y^2 - 3z^2 - 3t^2 = 0$. The total number of integral solutions of this equation in the range of the first 10000 numbers, i.e., $1 \leq x, y, z, t \leq 10000$, is

(a) 200

(b) 55

(c) 100

(d) 1

(e) None of the above

Part B
Computer Science Questions

1. Let T be a rooted binary tree whose vertices are labelled with symbols $a, b, c, d, e, f, g, h, i, j, k$. Suppose the in-order (visit left subtree, visit root, visit right subtree) and post-order (visit left subtree, visit right subtree, visit root) traversals of T produce the following sequences.

in-order: $a, b, c, d, e, f, g, h, i, j, k$

post-order: $a, c, b, e, f, h, j, k, i, g, d$

How many leaves does the tree have?

(a) THREE.

(b) FOUR.

(c) FIVE.

(d) SIX.

(e) Cannot be determined uniquely from the given information.

2. Consider the following code.

```
def brian(n):
    count = 0

    while ( n != 0 )
        n = n & ( n-1 )
        count = count + 1

    return count
```

Here `n` is meant to be an unsigned integer. The operator `&` considers its arguments in binary and computes their bit wise AND. For example, 22 `&` 15 gives 6, because the binary (say 8-bit) representation of 22 is 00010110 and the binary representation of 15 is 00001111, and the bit-wise AND of these binary strings is 00000110, which is the binary representation of 6.

What does the function `brian` return?

(a) The highest power of 2 dividing `n`, but zero if `n` is zero.

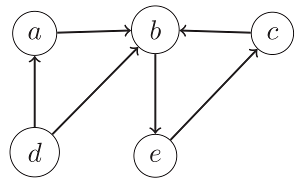
(b) The number obtained by complementing the binary representation of `n`.

(c) The number of ones in the binary representation of `n`.

(d) The code might go into an infinite loop for some `n`.

(e) The result depends on the number of bits used to store unsigned integers.

3. Consider the following directed graph.



Suppose a depth-first traversal of this graph is performed, assuming that whenever there is a choice, the vertex earlier in the alphabetical order is to be chosen. Suppose the number of tree edges is T, the number of back edges is B and the number of cross edges is C. Then

(a) $B = 1$, $C = 1$, and $T = 4$.

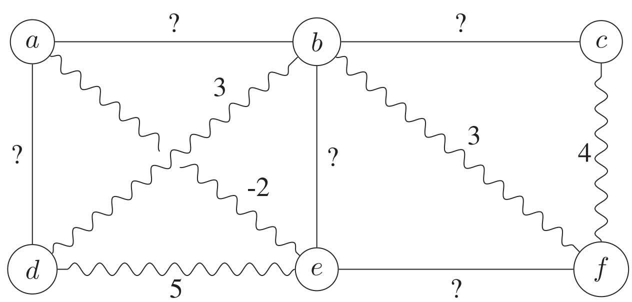
(b) $B = 0$, $C = 2$, and $T = 4$.

(c) $B = 2$, $C = 1$, and $T = 3$.

(d) $B = 1$, $C = 2$, and $T = 3$. ✓

(e) $B = 2$, $C = 2$, and $T = 1$.

4. Consider the following undirected graph with some edge costs missing.



Suppose the wavy edges form a *Minimum Cost Spanning Tree* for G. Then, which of the following inequalities NEED NOT hold?

(a) $\mathsf{cost}(a, b) \geq 6$. ✓

(b) $\mathsf{cost}(b, e) \geq 5$.

(c) $\mathsf{cost}(e, f) \geq 5$.

(d) $\mathsf{cost}(a, d) \geq 4$.

(e) $\mathsf{cost}(b, c) \geq 4$.

5. Let $G = (V, E)$ be an undirected connected simple (i.e., no parallel edges or self-loops) graph with the weight function $w : E \to \mathbb{R}$ on its edge set. Let $w(e_1) < w(e_2) < \cdots < w(e_m)$, where $E = \{e_1, e_2, \ldots, e_m\}$. Suppose T is a minimum spanning tree of G. Which of the following statements is FALSE?

(a) The tree T has to contain the edge e_1.

(b) The tree T has to contain the edge e_2.

(c) The minimum weight edge incident on each vertex has to be present in T.

(d) T is the unique minimum spanning tree in G.

(e) If we replace each edge weight $w_i = w(e_i)$ by its square w_i^2, then T must still be a minimum spanning tree of this new instance.

6. Consider the problem of computing the minimum of a set of n distinct numbers. We choose a permutation uniformly at random (i.e., each of the $n!$ permutations of $\langle 1, \ldots, n \rangle$ is chosen with probability $1/n!$) and we inspect the numbers in the order given by this permutation.

We maintain a variable MIN that holds the minimum value seen so far. MIN is initialized to ∞ and if we see a value smaller than MIN during our inspection, then MIN is updated. For example, in the inspection given by the following sequence, MIN is updated four times.

$\underline{5}$ 9 $\underline{4}$ $\underline{2}$ 6 8 $\underline{0}$ 3 1 7

What is the expected number of times MIN is updated?

(a) $O(1)$

(b) $H_n = \sum_{i=1}^{n} 1/i$

(c) $\sqrt{n}$

(d) $n/2$

(e) n

7. Which of the following statements is TRUE for all sufficiently large n?

(a) $(\log n)^{\log \log n} < 2^{\sqrt{\log n}} < n^{1/4}$.

(b) $2^{\sqrt{\log n}} < n^{1/4} < (\log n)^{\log \log n}$.

(c) $n^{1/4} < (\log n)^{\log \log n} < 2^{\sqrt{\log n}}$.

(d) $(\log n)^{\log \log n} < n^{1/4} < 2^{\sqrt{\log n}}$.

(e) $2^{\sqrt{\log n}} < (\log n)^{\log \log n} < n^{1/4}$.

8. Which of these functions grows fastest with n?

(a) e^n/n.

(b) $e^{n-0.9\log n}$.

(c) 2^n.

(d) $(\log n)^{n-1}$.

(e) None of the above.

9. Given a set of n distinct numbers, we would like to determine the smallest three numbers in this set using comparisons. Which of the following statements is TRUE?

(a) These three elements can be determined using $O(\log^2 n)$ comparisons.

(b) $O(\log^2 n)$ comparisons do not suffice, however these three elements can be determined using $n+O(1)$ comparisons.

(c) $n + O(1)$ comparisons do not suffice, however these three elements can be determined using $n + O(\log n)$ comparisons.

(d) $n+O(\log n)$ comparisons do not suffice, however these three elements can be determined using $O(n)$ comparisons.

(e) None of the above.

10. Given a set of n distinct numbers, we would like to determine *both* the smallest and the largest number. Which of the following statements is TRUE?

(a) These two elements can be determined using $O(\log^{100} n)$ comparisons.

(b) $O(\log^{100} n)$ comparisons do not suffice, however these two elements can be determined using $n + O(\log n)$ comparisons.

(c) $n+O(\log n)$ comparisons do not suffice, however these two elements can be determined using $3\lceil n/2\rceil$ comparisons.

(d) $3\lceil n/2\rceil$ comparisons do not suffice, however these two elements can be determined using $2(n-1)$ comparisons.

(e) None of the above.

11. Consider the following recurrence relation:

$$T(n) = \begin{cases} T(\frac{n}{k}) + T(\frac{3n}{4}) + n & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}$$

Which of the following statements is FALSE?

(a) $T(n)$ is $O(n^{3/2})$ when $k = 3$.

(b) $T(n)$ is $O(n \log n)$ when $k = 3$.

(c) $T(n)$ is $O(n \log n)$ when $k = 4$.

(d) $T(n)$ is $O(n \log n)$ when $k = 5$.

(e) $T(n)$ is $O(n)$ when $k = 5$.

12. Consider the following three statements:-
i) Intersection of infinitely many regular languages must be regular.
ii) Every subset of a regular language is regular.
iii) If L is regular and M is not regular then $L \cdot M$ is necessarily not regular.
Which of the following gives the correct true/false evaluation of the above?

(a) $true, false, true$.

(b) $false, false, true$.

(c) $true, false, true$.

(d) $false, false, false$.

(e) $true, true, true$.

13. Let L be a given context-free language over the alphabet $\{a, b\}$. Construct L_1, L_2 as follows. Let $L_1 = L - \{xyx \mid x, y \in \{a, b\}^*\}$, and $L_2 = L \cdot L$. Then,

(a) Both L_1 and L_2 are regular.

(b) Both L_1 and L_2 are context free but not necessarily regular.

(c) L_1 is regular and L_2 is context free.

(d) L_1 and L_2 both may not be context free.

(e) L_1 is regular but L_2 may not be context free.

14. Which the following is FALSE?

(a) Complement of a recursive language is recursive.

(b) A language recognized by a non-deterministic Turing machine can also be recognized by a deterministic Turing machine.

(c) Complement of a context free language can be recognized by a Turing machine.

(d) If a language and its complement are both recursively enumerable then it is recursive.

(e) Complement of a non-recursive language can never be recognized by any Turing machine.

15. Consider the set N^* of finite sequences of natural numbers with $x \leq_p y$ denoting that sequence x is a prefix of sequence y. Then, which of the following is true?

(a) N^* is uncountable.

(b) $\leq_p$ is a total order.

(c) Every non-empty subset of N^* has a least upper bound.

(d) Every non-empty subset of N^* has a greatest lower bound.

(e) Every non-empty finite subset of N^* has a least upper bound.

16. Consider the ordering relation $x \,|\, y \subseteq N \times N$ over natural numbers N such that $x|y$ iff there exists $z \in N$ such that $x \cdot z = y$. A set is called lattice if every finite subset has a least upper bound and greatest lower bound. It is called a complete lattice if every subset has a least upper bound and greatest lower bound. Then,

(a) $|$ is an equivalence relation.

(b) Every subset of N has an upper bound under $|$.

(c) $|$ is a total order.

(d) $(N, |)$ is a complete lattice.

(e) $(N, |)$ is a lattice but not a complete lattice.

17. Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function computed by a logical circuit comprising just binary AND and binary OR gates (assume that the circuit does not have any feedback). Let PARITY : $\{0,1\}^n \to \{0,1\}$ be the boolean function that outputs 1 iff the total number of input bits set to 1 is odd. Similarly, let MAJORITY be the boolean function that outputs 1 iff the number of input bits that are set to 1 is at least as large as the number of input bits that are set to 0. Then, which of the following is NOT possible?

(a) $f(0, 0, \ldots, 0) = f(1, 1, \ldots, 1) = 0.$

(b) $f(0, 0, \ldots, 0) = f(1, 1. \ldots, 1) = 1$

(c) f is the MAJORITY function.

(d) f is the PARITY function.

(e) f outputs 1 at exactly one assignment of the input bits.

18. Let k be an integer at least 4 and let $[k] = \{1, 2 \ldots, k\}$. Let $f : [k]^4 \to \{0,1\}$ be defined as follows: $f(y_1, y_2, y_3, y_4) = 1$ if an only if the y_i's are all distinct. For each choice $z = (z_1, z_2, z_3) \in [k]^3$, let $g_z : [k] \to \{0,1\}$ be defined by $g_z(Y) = f(Y, z_1, z_2, z_3)$. Let N be the number of distinct functions g_z that are obtained as z varies in $\{1, 2, \ldots, k\}^3$, that is, $N = |\{g_z : z \in \{1, 2, \ldots, k\}^3|\}$. What is N?

(a) $k^3 + 1$.

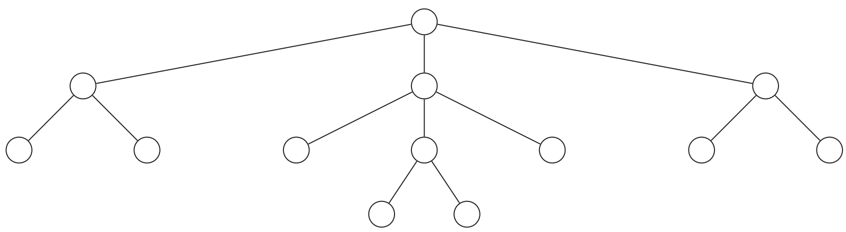
(b) $2^{\binom{k}{3}}$.

(c) $\binom{k}{3}$.

(d) $\binom{k}{3} + 1$. ✓

(e) $4\binom{k}{3}$.

19. Consider the following tree with 13 nodes.



Suppose the nodes of the tree are randomly assigned distinct labels from $\{1, 2, \ldots, 13\}$, each permutation being equally likely. What is the probability that the labels form a *min-heap* (i.e., every node receives the minimum label in its subtree)?

(a) $\left(\frac{1}{6!}\right)\left(\frac{1}{3!}\right)^2$

(b) $\left(\frac{1}{3!}\right)^2\left(\frac{1}{2!}\right)^3$

(d) $\left(\frac{1}{13}\right)\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)^3$ ✓

(c) $\frac{2}{13}$

(e) $\frac{1}{2^{13}}$

20. Consider the following game. There is a list of distinct numbers. At any round, a player arbitrarily chooses two numbers a, b from the list and generates a new number c by subtracting the smaller number from the larger one. The numbers a and b are put back in the list. If the number c is non-zero and is not yet in the list, c is added to the list. The player is allowed to play as many rounds as the player wants. The score of a player at the end is the size of the final list.

Suppose at the beginning of the game the list contains the following numbers: 48, 99, 120, 165 and 273. What is the score of the best player for this game?

(a) 40

(b) 16

(c) 33

(d) 91 ✓

(e) 123

Part C
Systems Science Questions

1. Consider two independent and identically distributed random variables X and Y uniformly distributed in $[0, 1]$. For $\alpha \in [0, 1]$, the probability that $\alpha \max(X, Y) < \min(X, Y)$ is

(a) $1/(2\alpha)$.

(b) $\exp(1 - \alpha)$

(c) $1 - \alpha$ ✓

(d) $(1 - \alpha)^2$

(e) $1 - \alpha^2$

2. Evaluate the limit

$$\lim_{n\to\infty} (2n^4)^{\frac{1}{3n}}.$$

(a) e

(b) 1 ✓

(c) $2^{\frac{1}{3}}$

(d) 0

(e) None of the above

3. For a non-negative continuous random variable X, which of the following is TRUE?

(a) $E\{X\} = \int_0^\infty P(X > x)dx$. ✓

(b) $E\{X\} = \int_0^\infty P(X \leq x)dx$.

(c) $P(X < x) \leq \frac{E\{X\}}{x}$.

(d) (a) and (c).

(e) None of the above.

4. A system accepts a sequence of real numbers $x[n]$ as input and outputs

$$y[n] = \begin{cases} 0.5x[n] - 0.25x[n-1], & n \text{ even} \\ 0.75x[n], & n \text{ odd} \end{cases}$$

The system is

(a) non-linear.

(b) non-causal.

(c) time-invariant.

(d) All of the above.

(e) None of the above. ✓

5. The matrix

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{pmatrix}$$

is invertible when

(a) $a_1 > a_2 > a_3$

(b) $a_1 < a_2 < a_3$

(c) $a_1 = 3$, $a_2 = 2$, $a_3 = 4$

(d) All of the above ✓

(e) None of the above

6. Let $g : [0, \pi] \to \mathbb{R}$ be continuous and satisfy

$$\int_0^\pi g(x) \sin(nx)\, dx = 0$$

for all integers $n \geq 2$. Then which of the following can you say about g?

(a) g must be identically zero.

(b) $g(\pi/2) = 1$.

(c) g need not be identically zero. ✓

(d) $g(\pi) = 0$.

(e) None of the above.

7. Let A be an $n \times n$ real matrix. It is known that there are two distinct n-dimensional real column vectors v_1, v_2 such that $Av_1 = Av_2$. Which of the following can we conclude about A?

(a) All eigenvalues of A are non-negative.

(b) A is not full rank. ✓

(c) A is not the zero matrix.

(d) $\det(A) \neq 0$.

(e) None of the above.

8. Consider a square pulse $g(t)$ of height 1 and width 1 centred at $1/2$. Define $f_n(t) = \frac{1}{n}\left(g(t) *^n g(t)\right)$, where $*^n$ stands for n-fold convolution. Let $f(t) = \lim_{n\to\infty} f_n(t)$. Then, which of the following is TRUE?

(a) The area under the curve of $f(t)$ is zero. ✓

(b) The area under the curve of $f(t)$ is ∞.

(c) $f(t)$ has width ∞ and height 1.

(d) $f(t)$ has width 0 and height ∞.

(e) None of the above.

9. Consider the following input $x(t)$ and output $y(t)$ pairs for two different systems.
(i) $x(t) = \sin(t)$, $y(t) = \cos(t)$,
(ii) $x(t) = t + \sin(t)$, $y(t) = 2t + \sin(t-1)$.
Which of these systems could possibly be linear and time invariant? Choose the most appropriate answer from below.

(a) (i), but not (ii). ✓

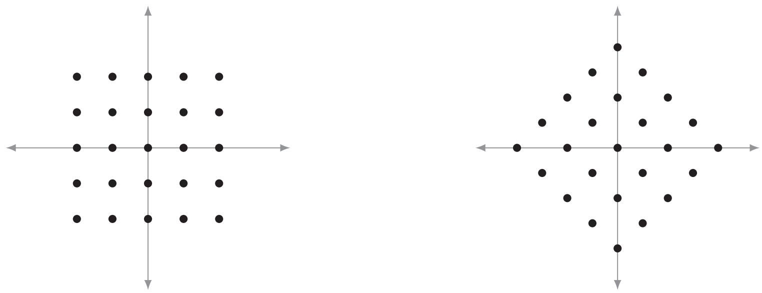
(b) (ii), but not (i).

(c) both (i) and (ii).

(d) neither (i) nor (ii).

(e) neither, but a system with $x(t) = \sin(2t)$, $y(t) = \sin(t)\cos(t)$ could be.

10. Consider the two quadrature amplitude modulation (QAM) constellations below. Suppose that the channel has additive white Gaussian noise channel and no intersymbol interference. The constellation points are picked equally likely. Let $P(\text{QAM})$ denote the average transmit power and $P_e(\text{QAM})$ denote the average probability of error under optimal decoding.



QAM_1 $\quad\quad$ QAM_2

Which of the following is TRUE?

(a) $P(\text{QAM}_1) > P(\text{QAM}_2)$, $P_e(\text{QAM}_1) < P_e(\text{QAM}_2)$.

(b) $P(\text{QAM}_1) < P(\text{QAM}_2)$, $P_e(\text{QAM}_1) > P_e(\text{QAM}_2)$.

(c) $P(\text{QAM}_1) > P(\text{QAM}_2)$, $P_e(\text{QAM}_1) > P_e(\text{QAM}_2)$.

(d) $P(\text{QAM}_1) < P(\text{QAM}_2)$, $P_e(\text{QAM}_1) < P_e(\text{QAM}_2)$.

(e) $P(\text{QAM}_1) = P(\text{QAM}_2)$, $P_e(\text{QAM}_1) = P_e(\text{QAM}_2)$.

11. It is known that the signal $x(t)$, where t denotes time, belongs to the following class:

$$\{A\sin(2\pi f_0 t + \theta) : \ f_0 = 1\text{Hz}, \ 0 \leq A \leq 1, \ 0 < \theta \leq \pi\}$$

If you are allowed to take samples of $x(t)$ at times of your choosing, at most how many samples are required to determine the signal?

(a) 1 sample.

(b) 2 samples.

(c) 1 sample per second.

(d) 2 samples per second.

(e) None of the above.

12. Assume that Y, Z are independent, zero-mean, continuous random variables with variances σ_Y^2 and σ_Z^2, respectively. Let $X = Y + Z$. The optimal value of α which minimizes $\mathbb{E}\left[(X - \alpha Y)^2\right]$ is

(a) $\frac{\sigma_Y^2}{\sigma_Y^2+\sigma_Z^2}$

(b) $\frac{\sigma_Z^2}{\sigma_Y^2+\sigma_Z^2}$

(c) 1 ✓

(d) $\frac{\sigma_Y^2}{\sigma_Z^2}$

(e) None of the above.

13. Let function $f : \mathbf{R} \to \mathbf{R}$ be convex, i.e., for $x, y \in \mathbf{R}$, $\alpha \in [0,1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$. Then which of the following is TRUE?

(a) $f(x) \leq f(y)$ whenever $x \leq y$.

(b) For a random variable X, $E(f(X)) \leq f(E(X))$.

(c) The second derivative of f can be negative.

(d) If two functions f and g are both convex, then $\min\{f, g\}$ is also convex.

(e) For a random variable X, $E(f(X)) \geq f(E(X))$. ✓

14. Suppose that a random variable X has a probability density function

$$\begin{aligned} f(x) &= c(x-4) \quad \text{for } 4 \leq x \leq 6 \\ &= 0 \quad \text{for all other } x \end{aligned}$$

for some constant c. What is the expected value of X given that $X \geq 5$?

(a) $5\frac{5}{9}$ ✓

(b) $5\frac{1}{2}$

(c) $5\frac{3}{4}$

(d) $5\frac{1}{4}$

(e) $5\frac{5}{8}$

15. You are allotted a rectangular room of a fixed height. You have decided to paint the three walls and put wallpaper on the fourth one. Walls can be painted at a cost of Rs. 10 per meter and the wall paper can be put at the rate of Rs 20 per meter for that specified height. What is the minimum cost at which you can achieve this covering for a 200 square meter room?

(a) $400 \times \sqrt{3}$ ✓

(b) 400

(c) $400 \times \sqrt{2}$

(d) $200 \times \sqrt{3}$

(e) 500

16. A fair dice (with faces numbered $1, \ldots, 6$) is independently rolled twice. Let X denote the maximum of the two outcomes. The expected value of X is

(a) $4\frac{1}{2}$

(b) $3\frac{1}{2}$

(c) 5

(d) $4\frac{17}{36}$ ✓

(e) $4\frac{3}{4}$

17. Let X be a Gaussian random variable with mean μ_1 and variance σ_1^2. Now, suppose that μ_1 itself is a random variable, which is also Gaussian distributed with mean μ_2 and variance σ_2^2. Then the distribution of X is

(a) Gaussian random variable with mean μ_2 and variance $\sigma_1^2 + \sigma_2^2$.

(b) Uniform with mean $\mu(\sigma_1^2 + \sigma_2^2)$.

(c) Gaussian random variable with mean μ_2 and variance $\sigma_1^2 + \sigma_2^2$. ✓

(d) Has no known form.

(e) None of the above.

18. A non-negative loss in a car accident is distributed with the following probability density function

$$f(x) = \frac{1}{10}\exp(-x/10)$$

for $x \geq 0$. Suppose that first 5 units of loss is incurred by the insured and the remaining loss if any is covered by the insurance company. What is the expected loss to the insurance company?

(a) $10\exp(-\frac{1}{2})$ ✓

(b) 10

(c) $5\exp(-\frac{1}{2})$

(d) $5 + 10\exp(-\frac{1}{2})$

(e) $15\exp(-\frac{1}{2})$

19. Consider a $2^k \times N$ binary matrix $A = \{a_{\ell,k}\},\ a_{\ell,k} \in \{0,1\}$. For rows i and j, let the Hamming distance be $d_{i,j} = \sum_{\ell=1}^{N} |a_{i,\ell} - a_{j,\ell}|$. Let $D_{min} = \min_{i,j} d_{i,j}$. Then which of the following is TRUE?

Hint: Consider any block of $k-1$ columns.

(a) $D_{min} \leq N - k + 1$. ✓

(b) $D_{min} \leq N - k$.

(c) $D_{min} \leq N - k - 1$.

(d) $D_{min} \leq N - k - 2$.

(e) None of the above.

20. What is

$$\lim_{n\to\infty} \cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^n}?$$

(a) 0

(b) $\pi/2$

(c) $1/\sqrt{2}$

(d) $2/\pi$ ✓

(e) None of the above.